ON HYPERBOLIC COMPLEX LTI DIGITAL SYSTEMS

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ABSTRACT

An investigation of digital systems based on hyperbolic (complex) numbers is presented for the first time. We consider the algebra of hyperbolic numbers (i.e. complex numbers possessing the imaginary $\mathrm{j}^2 = 1$*) as the simplest instance of a non-division algebra, which allows us to examine the impact of zero divisors in the case of DSP applications. As an example, the analysis of a general first order hyperbolic system is presented.*

1. INTRODUCTION

Complex numbers $\mathbb{C}$ have been in use for digital signal processing (DSP) purposes for a long time. They allow for compact signal and system descriptions and lead to differing system structures in comparison with real systems of the same function, occasionally resulting in a decreased computational expenditure. Furthermore, a general complex system has specific properties imposed by its algebraic structure, e.g. commutativity of cascaded complex sections, which is not self-evident if considered as a 2×2 LTI (Linear and Time-Invariant) MIMO (Multiple Input Multiple Output) system.

Hence, it is natural to ask about extending the complex number system to higher dimensional hypercomplex algebras. They comprise more than one imaginary unit and therefore establish n-dimensional (n-D), $n \in \mathbb{N}$, numbers in contrast to the 2-D complex numbers. Thereby, two major algebra classes can be distinguished: commutative and non-commutative ones. The latter are widely established in physics and have recently been shown to be useful for DSP applications, namely as the 4-D quaternions [1, 2] and other 2^N-D, $N \in \mathbb{N}$, Clifford algebras [2, 3]. No Clifford algebra except of the quaternions is a division algebra. This means that these algebras possess divisors of zero, for which division is lacking. Also the other major class, the commutative algebras, generally comprise zero divisors for $n > 2$, as Weierstrass has shown in 1883. In [4], he also proved that any commutative and associative algebra over the real numbers is isomorphic to a direct sum of the real numbers. We will examine this property in sec. 2.1, which can allow for efficient computation. However, so far only a few investigations are available for DSP application of commutative algebras [5, 6]. Frequently, the question of the impact of zero divisors on DSP is broached, but very rarely discussed satisfactorily.

In this paper, we select a pure example of a non-division algebra: the 2-D hyperbolic complex numbers $\mathbb{D}$, in the following briefly hyperbolic numbers (also referred to as "double number" [1, 7] or "split-complex number"). Every commutative and associative algebra (over $\mathbb{R}$) with $n > 2$ contains at least one hyperbolic subalgebra and therefore adopts its properties [4, 8]. Moreover, they are also incorporated into the Clifford algebra family [3]. In physics, they are related to Minkowsky spacetime [9]. They provide an extremely simple example of a non-division algebra and, therefore, we examine their properties with respect to DSP applications for the first time, thereby determining the impact of zero divisors regarding this purpose.

The outline of this paper is as follows: the hyperbolic numbers are briefly recalled (sec. 2), digital systems based on hyperbolic numbers are investigated (sec. 3), some generalisations are made (sec. 4), and finally the topic is concluded (sec. 5). We use the following notation: real numbers a, complex numbers $\underline{a}$, hyperbolic and other hypercomplex numbers $\boldsymbol{a}$, vectors $\mathbf{a}$ and matrices $\mathbf{A}$.

2. HYPERBOLIC NUMBERS

Corresponding to complex numbers $\underline{z} = x + \mathrm{i}y \in \mathbb{C}$, $x, y \in \mathbb{R}$, $\mathrm{i}^2 = -1$, a *hyperbolic number* [1, 3, 7, 9]

$$\boldsymbol{a} = a' + \mathrm{j}a'' \in \mathbb{D}, \quad a', a'' \in \mathbb{R}, \quad \mathrm{j}^2 = 1, \quad \mathrm{j} \notin \mathbb{R} \tag{1}$$

is composed of a real (a') and hyperbolic imaginary (a'') part. Obviously, addition of two hyperbolic numbers is performed componentwise: $\boldsymbol{a} + \boldsymbol{b} = a' + b' + \mathrm{j}\left(a'' + b''\right)$. In contrast to complex numbers, the square of the hyperbolic imaginary unit $\mathrm{j}^2 = 1$ is positive [10] (such imaginaries are also applied in Clifford algebras [2, 3]). Hence, the following multiplication rule results from (1):

$$\boldsymbol{ab} = \left(a' + \mathrm{j}a''\right)\left(b' + \mathrm{j}b''\right) = a'b' + a''b'' + \mathrm{j}\left(a'b'' + a''b'\right). \tag{2}$$

Corresponding to complex numbers, multiplication (2) is distributive over addition, *associative* and *commutative*. In conjunction with (2), hyperbolic numbers span a 2-D real vector space $\mathbb{R}^2$ with a specific multiplication rule. We define the hyperbolic conjugate

$$\overline{\boldsymbol{a}} = a' - \mathrm{j}a'', \quad \overline{\overline{\boldsymbol{a}}} = \boldsymbol{a}, \quad \overline{\boldsymbol{a} + \boldsymbol{b}} = \overline{\boldsymbol{a}} + \overline{\boldsymbol{b}}, \quad \overline{\boldsymbol{ab}} = \overline{\boldsymbol{a}} \cdot \overline{\boldsymbol{b}}, \tag{3}$$

with the same properties as the common complex conjugate, albeit valid only for hyperbolic numbers (cf. sec. 4.1). To separate the components of (1), the following operators are introduced applying (3):

$$\mathrm{Rh}\{\boldsymbol{a}\} := \frac{\boldsymbol{a} + \overline{\boldsymbol{a}}}{2} = a', \quad \mathrm{Ih}\{\boldsymbol{a}\} := \mathrm{j}\frac{\boldsymbol{a} - \overline{\boldsymbol{a}}}{2} = a'', \tag{4}$$

where $\mathrm{Rh}\{\boldsymbol{a}\}$ represents the real, and $\mathrm{Ih}\{\boldsymbol{a}\}$ the hyperbolic imaginary part of $\boldsymbol{a}$.

An *isomorphic* representation of a hyperbolic number $\boldsymbol{a}$, being completely equivalent to (1), is given by the real 2×2 matrix:

$$\mathbf{M}_{\boldsymbol{a}} = \begin{bmatrix} a' & a'' \\ a'' & a' \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \boldsymbol{a} = a' + \mathrm{j}a'' \in \mathbb{D}. \tag{5}$$

Any operation or property can likewise be performed using representation (1) or (5), respectively. For instance, it is readily shown that matrices of the form (5) are commutative regarding multiplication: $\mathbf{M}_{\boldsymbol{a}}\mathbf{M}_{\boldsymbol{b}} = \mathbf{M}_{\boldsymbol{b}}\mathbf{M}_{\boldsymbol{a}}$. However, calculations applying the matrix formulation are highly redundant and are predominantly applied for analytical purpose.

The modulus $|\underline{z}| = \sqrt{x^2 + y^2}$ of a complex number $\underline{z} \in \mathbb{C}$ is equal to its *Euclidean* distance from the origin. For hyperbolic numbers, however, a modulus $|\boldsymbol{a}| = \sqrt{a'^2 + a''^2}$ does not fit to the intrinsic nature of the hyperbolic number plane. For instance, in general the square identity [1] does not hold: $|\boldsymbol{ab}| \neq |\boldsymbol{a}|\,|\boldsymbol{b}|$. As a remedy, we define the *quadratic form* $N(\boldsymbol{a})$ of a hyperbolic number, deviating from $|\boldsymbol{a}|^2$, as follows:

$$N(\boldsymbol{a}) = \boldsymbol{a}\overline{\boldsymbol{a}} = a'^2 - a''^2 = \det \mathbf{M}_{\boldsymbol{a}} \in \mathbb{R}. \tag{6}$$

It will serve as a *norm*, but note that (6) can be negative. Nevertheless, basic matrix algebra ($\det \mathbf{A} \cdot \det \mathbf{B} = \det \mathbf{AB}$) applied to (5) and (6) shows that $N(\boldsymbol{a})$ satisfies the property $N(\boldsymbol{a})N(\boldsymbol{b}) = N(\boldsymbol{ab})$. In fig. 1, some constant contours of (6) are depicted in the hyperbolic number plane.

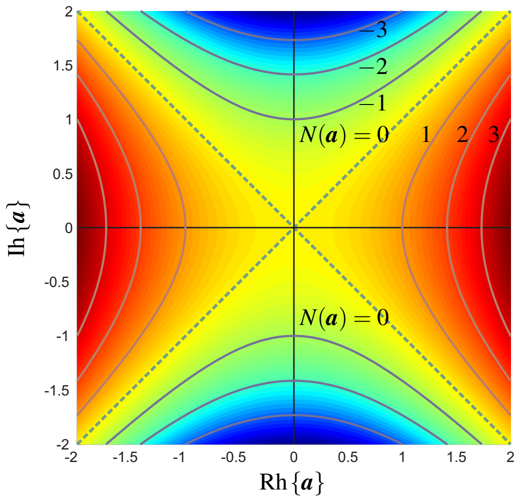


Figure 1: Contours of constant norm $N(\boldsymbol{a})$ (solid) in the hyperbolic plane with indication of the two zero divisor lines $N(\boldsymbol{a}) = 0$ (dashed)

2.1 Existence of zero divisors

As a direct consequence of $\mathrm{j}^2 = +1$, the hyperbolic numbers are not a *division* algebra. This means that the inverse of a hyperbolic number

$$\boldsymbol{a}^{-1} = \frac{\bar{\boldsymbol{a}}}{N(\boldsymbol{a})} = \frac{a' - \mathrm{j}a''}{a'^2 - a''^2} \tag{7}$$

is not always defined. Hence, although all other requirements are met, they do not form a *field* as complex numbers, but a commutative ring with unity. As a result, *zero divisors* [4] exist: a product of two non-zero numbers can yield zero, e.g. $(1+\mathrm{j})(1-\mathrm{j}) = 1-\mathrm{j}^2 = 0$. A zero divisor is not invertible, e.g. $(1+\mathrm{j})^{-1}$ (as well as 0^{-1} does not exist), and has a zero norm $N(\cdot)$. This is additionally confirmed with isomorphic matrix algebra (5), (6) allowing for inversion if and only if $\mathbf{M}_{\boldsymbol{a}}$ is non-singular ($\det \mathbf{M}_{\boldsymbol{a}} \neq 0$). Hence, all zero divisors in $\mathbb{D}$ are determined by

$$\det \mathbf{M}_{\boldsymbol{a}} = a'^2 - a''^2 = 0 \quad \Leftrightarrow \quad \left|a'\right| = \left|a''\right|,$$

forming two lines in the hyperbolic number plane (see fig. 1):

$$\boldsymbol{a}_1 = \alpha_1 (1+\mathrm{j}), \quad \boldsymbol{a}_2 = \alpha_2 (1-\mathrm{j}), \quad \alpha_1, \alpha_2 \in \mathbb{R}. \tag{8}$$

The product of any two numbers $\boldsymbol{a}_1$ times $\boldsymbol{a}_2$ results in zero.

2.2 Orthogonal representation

The existence of zero divisors allows for the *orthogonal decomposition* [3, 5, 9] of any hyperbolic number:

$$[\tilde{a}_1, \tilde{a}_2] = \tilde{a}_1 \boldsymbol{e}_1 + \tilde{a}_2 \boldsymbol{e}_2 = \tilde{a}_1 \frac{1+\mathrm{j}}{2} + \tilde{a}_2 \frac{1-\mathrm{j}}{2}, \quad \tilde{a}_1, \tilde{a}_2 \in \mathbb{R}. \tag{9}$$

This utilises an idempotent orthogonal system spanned by its base vectors $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{D}$, which fulfil the following properties [8]:

$$\boldsymbol{e}_1 \boldsymbol{e}_2 = 0, \quad \boldsymbol{e}_{1,2}^2 = \boldsymbol{e}_{1,2}, \quad \boldsymbol{e}_1 + \boldsymbol{e}_2 = 1 \in \mathbb{R}. \tag{10}$$

To derive the orthogonal base vectors $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{D}$, we apply the algebraic rule that every *idempotent* element $\boldsymbol{e}^2 = \boldsymbol{e}$ different from $0^2 = 0$ and $1^2 = 1$ (since $\boldsymbol{a}^2 - \boldsymbol{a} = \boldsymbol{a}(\boldsymbol{a}-1) = 0$) is a zero divisor:

$$\begin{aligned} a'^2 + 2\mathrm{j}a'a'' + a''^2 - a' - \mathrm{j}a'' &= 0 \\ \Rightarrow a''^2 = a' - a'^2 \quad \wedge \quad a'' &= 2a'a''. \end{aligned} \tag{11}$$

There exist two possible nontrivial ($\boldsymbol{a} \neq 0$) solutions of (11): $a'' = 0$ or $a' = \frac{1}{2}$. The former one leads to $0 = a' - a'^2 \Rightarrow a' = 1 \Rightarrow \boldsymbol{e} = 1$ (unit element 1), whereas the latter results in

$$a''^2 = \frac{1}{4} \Rightarrow a'' = \pm\frac{1}{2} \Rightarrow \boldsymbol{e}_1 = \frac{1+\mathrm{j}}{2}, \; \boldsymbol{e}_2 = \frac{1-\mathrm{j}}{2}, \tag{12}$$

the two idempotent zero divisors of $\mathbb{D}$.

In the orthogonal representation, any arithmetic operation, such as addition, multiplication, division or exponentiation, is performed componentwise [3, 9]:

$$\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{f}([\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2]) = [\boldsymbol{f}(\tilde{a}_1), \boldsymbol{f}(\tilde{a}_2)]. \tag{13}$$

Obviously, a calculation using the orthogonal representation saves computational complexity. For example, a hyperbolic multiplication according to (2), requires 4 real multiplications and 2 real additions, whereas it is reduced to only 2 real multiplications compliant with (13). Basically, this reflects that $\mathbb{D}$ is in fact not more than an isomorphism to the direct sum $\mathbb{R} \oplus \mathbb{R}$ (which results in the simple real vector space $\mathbb{R}^2$ without a separate multiplication rule) [4]. However, just because of this isomorphism we are able to lower the computational load of multiplication, while still realising the complete structure defined by (1) and (2), respectively. To enable this alternative, an *orthogonalisation*

$$\tilde{a}_1 = a' + a'', \quad \tilde{a}_2 = a' - a'' \tag{14}$$

and a *deorthogonalisation*

$$a' = \frac{1}{2}(\tilde{a}_1 + \tilde{a}_2), \quad a'' = \frac{1}{2}(\tilde{a}_1 - \tilde{a}_2). \tag{15}$$

procedure, following from (9), is needed.

3. HYPERBOLIC LTI SYSTEMS

In the following, we develop an approach to LTI digital systems based on the hyperbolic number system $\mathbb{D}$. The structure of such a system is determined by (2), and consists of two distinct real subsystems $H_{\mathrm{R}}(z)$ and $H_{\mathrm{H}}(z)$, both existing twice: fig. 2. As in the real and complex cases, a hyperbolic LTI system is completely specified by its (hyperbolic) impulse response $\boldsymbol{h}(k)$, which determines the relationship between input $\boldsymbol{x}(k)$ and output $\boldsymbol{y}(k)$:

$$\boldsymbol{y}(k) = \boldsymbol{h}(k) * \boldsymbol{x}(k), \quad k \in \mathbb{Z}, \quad \boldsymbol{h}(k), \boldsymbol{x}(k), \boldsymbol{y}(k) \in \mathbb{D}. \tag{16}$$

Thereby, the real-valued implementation of the convolution operator $*$ in (16) derives from the multiplication rule (2)

$$\boldsymbol{y}(k) = h'(k) * x'(k) + h''(k) * x''(k) + \mathrm{j}\left[h''(k) * x'(k) + h'(k) * x''(k)\right] \tag{17}$$

and therefore the hyperbolic convolution adopts its properties, such as commutativity.

3.1 Hyperbolic transfer functions

For frequency domain representation, we apply the z-transform. For a hyperbolic signal $\boldsymbol{x}(k)$ or impulse response $\boldsymbol{h}(k)$, the z-transform is carried out componentwise using the linearity property of $\mathscr{Z}\{\cdot\}$:

$$\boldsymbol{X}(z) = \mathscr{Z}\{\boldsymbol{x}(k)\} = \sum_{k=-\infty}^{\infty} x'(k) z^{-k} + \mathrm{j} \sum_{k=-\infty}^{\infty} x''(k) z^{-k} \in \mathbb{C} \otimes \mathbb{D}. \tag{18}$$

Due to the complex nature of the z-transform, $\boldsymbol{H}(z) = \mathscr{Z}\{\boldsymbol{h}(k)\}$ according to (18) is hyperbolic with complex components, in general: $\boldsymbol{H}(z) \in \mathbb{C} \otimes \mathbb{D}$ (tessarines [10], sec. 4.1). Thus, for hyperbolic systems (based on hyperbolic numbers $\mathbb{D}$ with real components) we have to deal with transfer functions which utilise a combination of two different number systems.

To separate the real and imaginary parts of a complex impulse response $\underline{h}(k) \in \mathbb{C}$ in the z-domain, commonly the operator [5, 11]

$$\mathrm{Ra}\{H(z)\} = \frac{H(z) + H^*(z^*)}{2} = \mathscr{Z}\left\{\frac{\underline{h}(k) + \underline{h}^*(k)}{2}\right\} \tag{19}$$

is applied. Note that the two independent subsystems of a hyperbolic system are real: $H_{\mathrm{R}}(z) = \mathrm{Ra}\{H_{\mathrm{R}}(z)\}$, $H_{\mathrm{H}}(z) = \mathrm{Ra}\{H_{\mathrm{H}}(z)\}$.

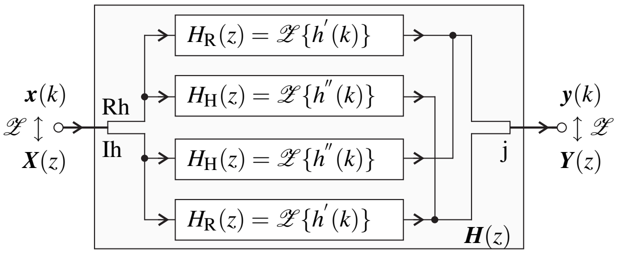


Figure 2: General hyperbolic system $\boldsymbol{H}(z)$, composed of two real subsystems $H_R(z)$ and $H_H(z)$

Hence, similar to (19), they can be separated from the transfer function $\boldsymbol{H}(z)$ of the hyperbolic system using (4):

$$H_R(z) = \text{Rh}\{\boldsymbol{H}(z)\} := \frac{\boldsymbol{H}(z)+\overline{\boldsymbol{H}}(\overline{z})}{2} = \frac{\boldsymbol{H}(z)+\overline{\boldsymbol{H}}(z)}{2}. \qquad (20)$$

$$H_H(z) = \text{Ih}\{\boldsymbol{H}(z)\} := \text{j}\frac{\boldsymbol{H}(z)-\overline{\boldsymbol{H}}(z)}{2}. \qquad (21)$$

Since hyperbolic conjugation $\overline{\{\cdot\}}$ has no effect on the complex compontents of tessarine numbers (cf. sec. 4.1), $\overline{z} = z$ in (20).

Another approach to a hyperbolic system is its representation as a real 2×2 MIMO system with vectorial input and output signals and impulse response, respectively:

$$\mathbf{x}(k) = \begin{bmatrix} x'(k) \\ x''(k) \end{bmatrix}, \quad \mathbf{y}(k) = \begin{bmatrix} y'(k) \\ y''(k) \end{bmatrix}, \quad \mathbf{h}(k) = \begin{bmatrix} h'(k) \\ h''(k) \end{bmatrix}.$$

In this case, the corresponding transfer matrix can readily be derived from (5):

$$\mathbf{H}(z) = \begin{bmatrix} H_R(z) & H_H(z) \\ H_H(z) & H_R(z) \end{bmatrix} = \begin{bmatrix} \text{Rh}\{\boldsymbol{H}(z)\} & \text{Ih}\{\boldsymbol{H}(z)\} \\ \text{Ih}\{\boldsymbol{H}(z)\} & \text{Rh}\{\boldsymbol{H}(z)\} \end{bmatrix}. \qquad (22)$$

However, note that for general hypercomplex algebras it is not always possible to derive the corresponding real MIMO transfer matrix directly from the algebra isomorphism matrix [8]. Here this is feasible because (5) contains the component vector $\begin{bmatrix} a' & a'' \end{bmatrix}^{\text{T}}$ in the first column. Otherwise we had to apply the algebra's multiplication rule. For hyperbolic systems, we confirm (22) using (2).

Since hyperbolic numbers are commutative, the basic convolution theorem, linking (16) and (18), holds [6, 12]:

$$\boldsymbol{y}(k) = \boldsymbol{h}(k) * \boldsymbol{x}(k) \;\leftrightarrow\; \boldsymbol{Y}(z) = \boldsymbol{H}(z)\boldsymbol{X}(z). \qquad (23)$$

3.2 Orthogonal decomposition

We do not have to implement the hyperbolic system according to (17), or (22), respectively, because the underlying algebra allows for orthogonal decomposition (as a result of the existence of zero divisors). Hence, a minimal system realisation is based on (9), and signal processing is performed according to (13), as depicted in fig. 3: the hyperbolic signal is fed into the *orthogonaliser* $\mathbf{F}$, parallelly processed by the (only) two orthogonal real subsystems $\tilde{H}_1(z)$ and $\tilde{H}_2(z)$, and finally retrieved from the *deorthogonaliser* $\mathbf{E}$. The 2×2 real MIMO representation of this processing chain is given by

$$\mathbf{H}(z) = \mathbf{E}\cdot\tilde{\mathbf{H}}(z)\cdot\mathbf{F}, \qquad (24)$$

which utilises the orthogonaliser

$$\tilde{\mathbf{x}}(k) = \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x'(k) \\ x''(k) \end{bmatrix} = \mathbf{F}\mathbf{x}(k), \qquad (25)$$

following (14), and the deorthogonaliser

$$\mathbf{x}(k) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} = \mathbf{E}\tilde{\mathbf{x}}(k) = \frac{1}{2}\mathbf{F}\tilde{\mathbf{x}}(k), \qquad (26)$$

derived from (9). The MIMO transfer matrix of the orthogonalised system according to (24)

$$\tilde{\mathbf{H}}(z) = \begin{bmatrix} \tilde{H}_1(z) & 0 \\ 0 & \tilde{H}_2(z) \end{bmatrix}, \quad \boldsymbol{H}(z) = \left[\tilde{H}_1(z), \tilde{H}_2(z)\right] \qquad (27)$$

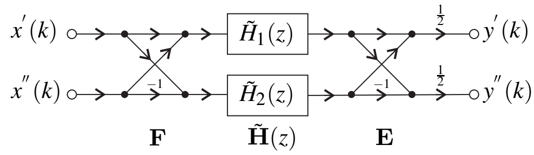


Figure 3: Real 2×2 MIMO representation of the processing chain utilising the orthogonal decomposition

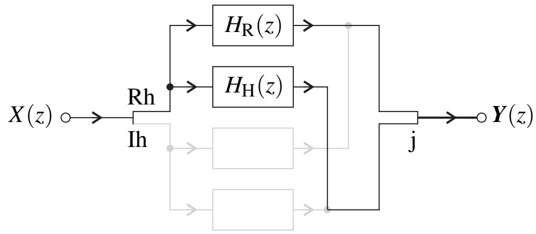


Figure 4: General hyperbolic system in the case of a real input

is diagonal and, therefore, reflects the componentwise processing of the orthogonalised signal. Using (14), (20) and (21), the subsystems of (27) are related to the subsystems of (22) as follows: $\tilde{H}_1(z) = H_R + H_H$, $\tilde{H}_2 = H_R - H_H$.

3.3 Hyperbolic processing of real signals

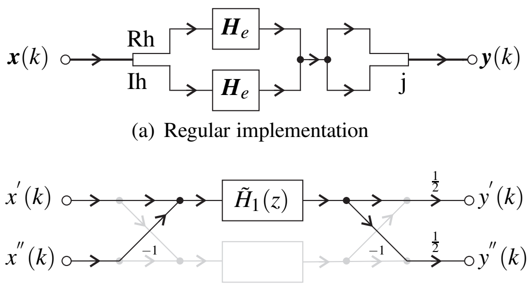
The main difference between the processing of purely real vectorial signals and the processing of (hyper)complex (e. g. hyperbolic) signals results from the underlying specific multiplication rules. It is obvious from fig. 2 that a hyperbolic *input signal* is subjected to a processing in *four* subsystems in compliance with (17). In contrast, in $\boldsymbol{H}(z) = \mathscr{Z}\{\boldsymbol{h}(k)\}$ the z-transform (18) has only been applied to the *two* components of the hyperbolic impulse response. To match these two apparently contradictory statements, firstly, it should be noted that the hyperbolic impulse response $\boldsymbol{h}(k)$ is the result of an excitation of the system by (17) with the *real* unit impulse

$$\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}.$$

Therefore, for all $k \in \mathbb{Z}$, the hyperbolic system is in fact reduced to two subsystems: fig. 4. Hence, the proposition that the hyperbolic system is completely described by its impulse response is true only if we imply the particular hyperbolic system structure, as given in fig. 2 or (17), respectively. Otherwise, we had to treat the system as a general 2×2 MIMO system and to describe it with an impulse *matrix*. Finally, since we transform this "pruned" impulse response $\boldsymbol{h}(k)$ according to (18) to obtain $\boldsymbol{H}(z)$, we account for the case depicted in fig. 4 once again. Yet, based on the structural knowledge inherent in (17), this description is complete and immediately calls for the overall structure depicted in fig. 2. As a result of this consideration, likewise applicable to complex systems, hypercomplex (e. g. hyperbolic) signal processing must not be treated vectorially, whether or not the z-transform is applied in a vectorial manner.

Next, we consider the processing of a general *real* signal by a *nonrecursive* hyperbolic system. In case of a compact implementation of the system according to fig. 2, the system is in fact "degenerated" to the structure shown in fig. 4. Such a hyperbolic system can, for instance, be applied as an FIR analysis (or its transpose as a synthesis) filter bank, with one (two) real input signal and two (one) real output signals.

Furthermore, we consider for the same application a *recursive* hyperbolic system. Here, the hyperbolic output signal is recursively fed back to the system input ports. Hence, the hyperbolic system requires "full" realisation, as depicted in fig. 2. We can obtain a similar result if the hyperbolic system is realised as a *cascade* of (recursive or nonrecursive) hyperbolic subsystems, which immediately follows from the overall real 2×2 MIMO transfer matrix of



(a) Regular implementation

(b) Orthogonal implementation

Figure 5: Hyperbolic system at the frequency Ω_e for which the system's frequency response $\boldsymbol{H}\left(\mathrm{e}^{\mathrm{j}\Omega_e}\right)$ is a zero divisor

the cascade of two hyperbolic systems $\boldsymbol{H}_1(z)$ and $\boldsymbol{H}_2(z)$:

$$\mathbf{H}(z) = \mathbf{H}_2(z)\cdot\mathbf{H}_1(z) = \begin{bmatrix} H_{2\mathrm{R}}(z) & H_{2\mathrm{H}}(z) \\ H_{2\mathrm{H}}(z) & H_{2\mathrm{R}}(z) \end{bmatrix} \cdot \begin{bmatrix} H_{1\mathrm{R}}(z) & H_{1\mathrm{H}}(z) \\ H_{1\mathrm{H}}(z) & H_{1\mathrm{R}}(z) \end{bmatrix}$$

$$= \begin{bmatrix} H_{1\mathrm{R}}(z)H_{2\mathrm{R}}(z)+H_{1\mathrm{H}}(z)H_{2\mathrm{H}}(z) & H_{1\mathrm{H}}(z)H_{2\mathrm{R}}(z)+H_{1\mathrm{R}}(z)H_{2\mathrm{H}}(z) \\ H_{1\mathrm{H}}(z)H_{2\mathrm{R}}(z)+H_{1\mathrm{R}}(z)H_{2\mathrm{H}}(z) & H_{1\mathrm{R}}(z)H_{2\mathrm{R}}(z)+H_{1\mathrm{H}}(z)H_{2\mathrm{H}}(z) \end{bmatrix}.$$

Obviously, $\mathrm{Rh}\{\boldsymbol{H}(z)\}$ and $\mathrm{Ih}\{\boldsymbol{H}(z)\}$ of the overall system comprise the transfer functions of all subsystems. Moreover, the output signal of the first cascade's stage is generally a hyperbolic signal, allowing for the first stage a "pruned" implementation according to fig. 4, while all subsequent stages are always of the form according to fig. 2, independently of the (non)recursive nature of the overall system.

Finally, we consider the processing of a general real signal by a *recursive* hyperbolic system, of which only its real input and output ports are used, i. e. we have a *SISO* (single input single output) system. Such an arrangement can be used to implement a real system's transfer function employing a hyperbolic system: Cf. sec. 3.5. Despite the fact that in the above cases we take advantage of the full hyperbolic structures, we are able to lower the computational load almost by a factor of two by applying the orthogonal decomposition.

3.4 Impact of zero divisors

The existence of zero divisors does not only influence the feasibility of division (7), which is not needed for DSP in general, but likewise affects the invertibility of hyperbolic LTI systems. For many applications, for instance perfect reconstruction filter banks, it is crucial whether or not an operation is invertible. In the following, we examine the case where a given hyperbolic frequency response $\boldsymbol{H}(z)|_{z=\mathrm{e}^{\mathrm{j}\Omega}} = \boldsymbol{H}\left(\mathrm{e}^{\mathrm{j}\Omega}\right)$ exhibits a zero divisor value $\boldsymbol{H}\left(\mathrm{e}^{\mathrm{j}\Omega_e}\right)$ at a particular normalised frequency Ω_e. Furthermore, we assume that the zero divisor value belongs to the following zero divisor line of (8): $\boldsymbol{H}\left(\mathrm{e}^{\mathrm{j}\Omega_e}\right) = \alpha(1+\mathrm{j})$, $\alpha \in \mathbb{R}$. According to (20) and (21), the two subsystems of the hyperbolic system are equal for this particular frequency point: $\mathrm{Rh}\left\{\boldsymbol{H}\left(\mathrm{e}^{\mathrm{j}\Omega_e}\right)\right\} = \mathrm{Ih}\left\{\boldsymbol{H}\left(\mathrm{e}^{\mathrm{j}\Omega_e}\right)\right\} = \boldsymbol{H}_e$. As a consequence, information is lost, because we have in fact only one distinguishable real signal at the output of the hyperbolic system. A compact implementation of such a degenerated system (only valid for Ω_e!) is depicted in fig. 5(a). In a corresponding orthogonal structure, the frequency response of the second orthogonal subsystem vanishes: $\tilde{H}_2\left(\mathrm{e}^{\mathrm{j}\Omega_e}\right) = 0$. This results from even another definition of a zero divisor (additional to the identification in sec. 2), which states that for a zero divisor, at least one of the orthogonal components is zero. Therefore, we have a blocking in the hyperbolic system which again results in a loss of information: fig. 5(b).

3.5 General first order hyperbolic LTI system

In the following, a general recursive hyperbolic LTI system of first order (fig. 6), implementing the difference equation

$$\boldsymbol{y}(k)+\boldsymbol{a}_1\boldsymbol{y}(k-1) = \boldsymbol{b}_0\boldsymbol{x}(k)+\boldsymbol{b}_1\boldsymbol{x}(k-1), \tag{28}$$

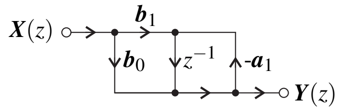


Figure 6: Hyperbolic LTI system of first order

is analysed, resulting in the hyperbolic transfer function

$$\boldsymbol{H}(z) = \frac{\boldsymbol{b}_0+\boldsymbol{b}_1 z^{-1}}{1+\boldsymbol{a}_1 z^{-1}}, \quad \boldsymbol{a}_1, \boldsymbol{b}_0, \boldsymbol{b}_1 \in \mathbb{D}. \tag{29}$$

Each coefficient is a hyperbolic number and can be split into two distinct real-valued parts, e. g. $\boldsymbol{b}_0 = b_0^{'} + \mathrm{j}b_0^{''}$, $b_0^{'}, b_0^{''} \in \mathbb{R}$. Using (20) and $\boldsymbol{a}\overline{\boldsymbol{b}}+\overline{\boldsymbol{a}}\boldsymbol{b} = 2\left(a^{'}b^{'}-a^{''}b^{''}\right)$, the transfer function of the first subsystem is given by:

$$\begin{aligned} H_{\mathrm{R}}(z) &= \frac{1}{2}\left(\frac{\boldsymbol{b}_0+\boldsymbol{b}_1 z^{-1}}{1+\boldsymbol{a}_1 z^{-1}} + \frac{\overline{\boldsymbol{b}}_0+\overline{\boldsymbol{b}}_1 z^{-1}}{1+\overline{\boldsymbol{a}}_1 z^{-1}}\right) \\ &= \frac{b_0^{'}+\left[b_1^{'}+a_1^{'}b_0^{'}-a_1^{''}b_0^{''}\right]z^{-1}+\left(a_1^{'}b_1^{'}-a_1^{''}b_1^{''}\right)z^{-2}}{1+2a_1^{'}z^{-1}+\left(a_1^{'2}-a_1^{''2}\right)z^{-2}}. \end{aligned} \tag{30}$$

Applying (21) and $\boldsymbol{a}\overline{\boldsymbol{b}}-\overline{\boldsymbol{a}}\boldsymbol{b} = 2\mathrm{j}\left(a^{''}b^{'}-a^{'}b^{''}\right)$, we derive the second subsystem's transfer function:

$$\begin{aligned} H_{\mathrm{H}}(z) &= \frac{\mathrm{j}}{2}\left(\frac{\boldsymbol{b}_0+\boldsymbol{b}_1 z^{-1}}{1+\boldsymbol{a}_1 z^{-1}} - \frac{\overline{\boldsymbol{b}}_0+\overline{\boldsymbol{b}}_1 z^{-1}}{1+\overline{\boldsymbol{a}}_1 z^{-1}}\right) \\ &= \frac{b_0^{''}+\left[b_1^{''}+a_1^{'}b_0^{''}-a_1^{''}b_0^{'}\right]z^{-1}+\left(a_1^{'}b_1^{''}-a_1^{''}b_1^{'}\right)z^{-2}}{1+2a_1^{'}z^{-1}+\left(a_1^{'2}-a_1^{''2}\right)z^{-2}}. \end{aligned} \tag{31}$$

We see that the degree of the real subsystems (30) and (31) is doubled relative to the hyperbolic system's degree. Hence, if we want to realise a real rational transfer function of a particular degree, we may employ a hyperbolic system of half degree (although considering that a hyperbolic delay is implemented by two real delays). The same relationship also applies for complex [13] and some other hypercomplex systems [5]. The system can also be implemented according to fig. 3. For the FIR case $\boldsymbol{a}_1 \equiv 0$, we see that the z^{-2} terms vanish in the numerators of (30) and (31), which means that the hyperbolic system (29) is reduced to a purely vectorial system (fig. 4) of only first order.

As an *example*, we realise the second order real system

$$H_{\mathrm{prot}}(z) = \frac{b_0+b_1 z^{-1}+b_2 z^{-2}}{1+a_1 z^{-1}+a_2 z^{-2}}, \tag{32}$$

$a_1 = -0.5095$, $a_2 = -0.1334$, $b_0 = b_2 = 0.4320$, $b_1 = -0.5078$, as a first order hyperbolic system according to (29). The magnitude response of the real prototype (32), a band stop filter with the normalised notch frequency $\Omega = 0.3\pi$, is shown in fig. 7. Arbitrarily, we select $H_{\mathrm{R}}(z)$ to comply with $H_{\mathrm{prot}}(z)$. Comparing coefficients of (30) and (32) yields:

$$a_1^{'} = \frac{1}{2}a_1 = -0.2547,\ a_1^{''} = \sqrt{\frac{1}{4}a_1^2 - a_2} = 0.4452 \tag{33}$$

$$b_0^{'} = b_0,\ b_0^{''} = 0,\ b_1^{'} = b_1 - \frac{1}{2}a_1 b_0 = -0.3978, \tag{34}$$

$$b_1^{''} = \frac{a_1 b_1 - \frac{1}{2}a_1^2 b_0 - 2b_2}{2a_1^{''}} = -0.7426. \tag{35}$$

In fig. 7 the magnitude responses of the two subsystems (30) and (31) are depicted, of which the former equals the prototype filter response, as required, and the latter is not used.

The same result is obtained by orthogonal implementation according to (24) and fig. 3, respectively, discarding unused connections in (25) and (26). Applying (14) to (33)-(35), the coefficients of the first order real orthogonal subsystems $\tilde{H}_1(z)$ and $\tilde{H}_2(z)$ in (27) are: $\tilde{a}_{1,1} = 0.1905$, $\tilde{b}_{0,1} = 0.4320$, $\tilde{b}_{1,1} = -1.1404$, $\tilde{a}_{1,2} = -0.7000$,

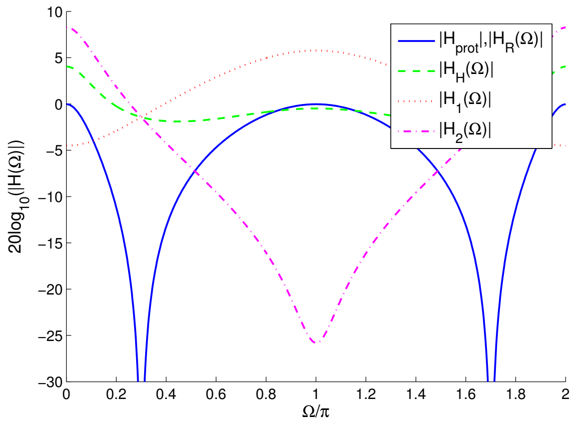


Figure 7: Magnitude response (in dB) of prototype filter $H_{\text{prot}}\left(e^{j\Omega}\right)$, subsystems $H_R\left(e^{j\Omega}\right)$ and $H_H\left(e^{j\Omega}\right)$ of hyperbolic filter, and corresponding orthogonal subsystems $\tilde{H}_1\left(e^{j\Omega}\right)$ and $\tilde{H}_2\left(e^{j\Omega}\right)$

$\tilde{b}_{0,2} = 0.4320$, $\tilde{b}_{1,2} = 0.3448$. The frequency response of the complete system (24) is identical to that of the implementation (22). In fig. 7, also the magnitude responses of the orthogonal subsystems $\tilde{H}_1(z)$ and $\tilde{H}_2(z)$ are shown. Since $\left|\tilde{H}_{1,2}\left(e^{j\Omega}\right)\right| > 0$ for all Ω, no zero divisors occur in the transfer function $\boldsymbol{H}(z)$ of the examined hyperbolic system.

4. GENERALISATIONS

4.1 Hyperbolic numbers with complex components

Like real numbers, hyperbolic numbers are algebraically not closed under exponentiation. For instance, the square root $\sqrt{\cdot} = (\cdot)^{\frac{1}{2}}$ in (33) may become imaginary with certain prototype filter coefficients. Hence, not every real transfer function can be realised applying the method presented in sec. 3.5. Furthermore, the complex kernel of the z-transform (18) leads to transfer functions both hyperbolic and complex. Therefore, it is straightforward to permit complex components in hyperbolic numbers (1), resulting in *tessarines* [8, 10]:

$$\boldsymbol{a} = \underline{a}' + j\underline{a}'' \in \mathbb{C} \otimes \mathbb{D}, \quad \underline{a}', \underline{a}'' \in \mathbb{C}, \quad j^2 = 1.$$

For tessarines, both hyperbolic conjugation $\overline{\{\cdot\}}$ and complex conjugation $\{\cdot\}^*$ are defined, and are clearly distinguiashable: $\overline{\boldsymbol{a}} = \underline{a}' - j\underline{a}''$ and $\boldsymbol{a}^* = \underline{a}'^* + j\underline{a}''^*$.

A tessarine LTI system [6, 8] consists of 16 (4 different) real subsystems and is, similar to the complex and hyperbolic cases (sec. 3.5), capable of doubling the degree of a hyperbolic system once more. However, it is not possible to reduce the expenditure of complex coefficient multiplication by further orthogonalisation. Hence, also for tessarine systems, the only feasible orthogonal decomposition is represented by (14), (15), (25) and (26).

4.2 2^N-dimensional hyperbolic numbers

In order to process vector signals of higher dimension n, or to increase the degree multiplication (sec. 3.5), it is possible to define hyperbolic numbers with hyperbolic components. By repeating the doubling procedure [1], numbers of dimension $n = 2^N$, $N \in \mathbb{N}$ can be obtained [14]. We stress that, in contrast to the step from real to hyperbolic numbers, further increase of dimension does not produce any additional issue. Every property of hyperbolic numbers is retained even for higher dimensions. Orthogonalisation can be extended to an n-times decomposition. It can be performed efficiently by the (fast) Hadamard transform [8].

The structure depicted in fig. 3 is in fact equal to common coupled, e. g. allpass systems. It is an open question, if it is feasible to employ higher dimensional hyperbolic LTI systems to describe coupled systems with more than two input and output ports, in order to simplify the design of such. The 2^N-dimensional approach can also be combined with the use of complex components (sec. 4.1).

5. CONCLUSION

A first step was made to describe the properties and efficient implementation of hyperbolic digital systems in general. Since hyperbolic subalgebras emerge in many other non-division hypercomplex algebras, they represent a comprehensible example for the examination of the impact of zero divisors on DSP. However, hyperbolic systems certainly are most useful if the underlying structure is somehow related to the application aimed at. Presently, due to limitations of realisable transfer functions, they do not represent a self-contained system class. Nevertheless, further investigation has to reveal if structures, as proposed in sec. 3.5, are feasible and efficient even for general DSP purposes.

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